LECTURE 37 THE FUNDAMENTAL THEOREM OF CALCULUS

Average Value of A Function

We first show some examples about average values of functions. In layman's term, recall that the classical mean value theorem states that your average speed must be achieved at least one time during your trip. Now, remember, average speed has unit length/time, and the classical mean value theorem is drawing a parallel between two speeds, one average, one instantaneous.

Can we, however, say a similar statement about a quantity that involves length only, say, displacement? Suppose $x(t)$ meausures displacement at time t, and we travel from $t = a$ to $t = b$. We would have achieved some average displacement, given by definition,

$$ave\, displacement = \frac{1}{b-a}\int_a^b x(t)\, dt.$$

Does it make sense that this average displacement is in fact achieved by some the displacement at some particular time, say, $x(c)$ for $c \in [a,b]$? In other words, your average value of a function $f(x)$ is a number J, and thus has a graph of a flat line $g(x) = J$. Will this line intersect $f(x)$? Let's draw a picture.

Theorem. *(Mean Value Theorem for Definite Integrals) If f is a continuous function, then at some point $c \in [a,b]$, we have*

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

Remark. The hypothesis that f is continuous is very important. A discontinuous function need not assume its average value. Consider

$$f(x) = \begin{cases} 0, & 0 \le x \le 1, \\ 1, & 1 < x \le 2, \end{cases}$$

which clearly has an average value of $\frac{1}{2}$, yet this number is never achieved by any function values.

HW: Compute the following definite integrals using the definition of a Riemann sum and then take a limit as $n \to \infty$ where n is the number of subintervals. Use whichever endpoint rule you want.

(1)
$$\int_a^b c dx$$
where c is a constant. This problem does not need a Rieman sum. It's simple geometry.

(2)
$$\int_a^b x dx.$$
For this, you also don't need Riemann sum but simple geometry. But feel free to confirm using a Riemann sum.

(3)
$$\int_a^b x^2 dx.$$

(4) Together using 1-3, can you express
$$\int_a^b \left(c_1x^2 + c_2x + c_3\right) dx$$
in terms of a, b, c_1, c_2 and c_3?

(5) Confirm that the function $f(x) = 9x^2 - 16x + 4$ does achieve its average value on $[0,2]$ via the following steps:

(a) Compute the average value of $f(x)$ via

$$f_{ave} = \frac{1}{2-0}\int_0^2 f(x)\,dx$$

using part 4.

(b) Use intermediate value theorem to confirm that $f(c) = f_{ave}$ for some number $c \in [0,2]$, at least once.

The Fundamental Theorem of Calculus Part I

Last lecture, we defined the definite integral

$$\int_a^b f(x)\,dx$$

as the limit value of the Riemann sum,

$$J = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

where $\|P\| = \max_k(\Delta x_k)$, the maximal subinterval length. However, we did NOT study how to evaluate a definite integral given $f(x)$ and $[a,b]$. We do not have a systematic way of computing $\int_a^b f(x)\,dx$ other than computing the limiting value of a Riemann sum. In this section, we provide a powerful tool that links between definite integrals and antiderivatives. The argument relies on a beautiful visualisation of the definite integral.

Consider the function $F(x)$ defined by

$$F(x) = \int_a^x f(t)\,dt,$$

that is, $F(x)$ is the area under $f(t)$ from a to x, where $f(t)$ is a continuous function. Note that now we can toggle x, which effectively changes the value of F by obtaining a different area under $f(t)$.

Remark. Question: Why such a contrived form of $F(x)$? Why do we even care?

Answer: If we know the function form of $F(x)$ given $f(t)$, then we know the area $\int_a^x f(t)\,dt$ for any x **without** computing Riemann sums.

Remark. Question: If $F(x)$ a differentiable function? Why do we care about whether F is a differentiable or not?

Answer: We check the limit of the difference quotient, that is,

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}.$$

Since we want the function form of $F(x)$, it would be nice to know its slope information. If it is differentiable, then we can nail down $F'(x)$ for every x and thus $F(x)$ is merely one of its antiderivative. We can then also relate $F'(x)$ to the integrand $f(t)$ (in a way we are yet to see until we compute it).

We go on to compute each quantity in the difference quotient.

$$F(x+h) = \int_a^{x+h} f(t)\,dt$$
$$F(x) = \int_a^x f(t)\,dt$$

Thus, using the property of the definite integral (you can patch things up, see Property 4 from last lecture note),

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} \overset{\text{Property 4}}{=} \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt$$

Now, suppose h is really really small (since it is going to 0 anyways), then how do we picture $\int_x^{x+h} f(t)\,dt$? It is a thin strip of a rectangle, with side length h and height $f(x)$. In addition to the heuristics, we can use the Mean Value Theorem for definite integrals here (check the hypothesis, we have $a = x$, $b = x + h$ and $f(t)$ is continuous). There must exist some number $c \in [x, x+h]$ such that

$$f(c) = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

Now, we have

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} f(c).$$

Here is where the continuity of f comes in. As we shrink $h \to 0$, we are shrinking the interval $[x, x+h]$ into a singleton x. Therefore,

$$\lim_{h\to 0} f(c) = f(x),$$

via the continuity of f (the limit of a function equals its function value). Therefore, the limit

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

exists, and by definition of the derivative, the LHS is $F'(x)$. Altogether, we have proved

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

This is to say that F is **an** antiderivative of f.

The Fundamental Theorem of Calculus Part II

Now, you may say, cool, we just defined a new function $F(x) = \int_a^x f(t)\,dt$ and we found out F is the antiderivative of f. So what? Is it going to help me evaluate definite integrals with a less messy method than doing Riemann sums? Yes.

First, remember that the antiderivative of a continuous function $f(x)$ is NOT unique. That is, if $F(x)$ is a known antiderivative, then so is $G(x) = F(x) + C$. Suppose now we have two candidate antiderivatives F and G defined as above. Let's compute

$$\begin{aligned} G(b) - G(a) &= F(b) + C - F(a) - C \\ &= F(b) - F(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt \\ &= \int_a^b f(t)\,dt \end{aligned}$$

What we found here is that the definite integral

$$\int_a^b f(t)\,dt$$

only depends on the value difference of **one** antiderivative. This means, as long as you find **any** antiderivative $G(x)$, you evaluate $G(b) - G(a)$ and this difference gives you the definite integral. No more Riemann sums. We sometimes write

$$G(b) - G(a) = [G(x)]_{x=a}^{x=b}$$

or

$$G(b) - G(a) = G(x)\mid_{x=a}^{x=b} = G(x)\mid_a^b.$$

Example. Compute the following definite integrals:

(1) $\int_0^\pi \sin(x)\,dx = [-\cos(x)]_0^\pi = -\cos(\pi) - (-\cos(0)) = 1 - (-1) = 2.$

(2) $\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = \frac{3}{2}\int_1^4 x^{\frac{1}{2}}dx - 4\int_1^4 x^{-2}dx = \frac{3}{2}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4 - 4\left[-x^{-1}\right]_1^4 = \left[4^{\frac{3}{2}} - 1\right] - 4\left[-\frac{1}{4} - (-1)\right] = 7 - 4 \cdot \frac{3}{4} = 4.$

Remark. It is absolutely crucial that you understand how integrals, derivatives and antiderivatives are related, and it is NOT obvious. The essence of their relationship is based on limits first, then some properties of the definite integral, and lastly, some properties of antiderivatives.

Corollary.

$$F(b) - F(a) = \int_a^b F'(x)\,dx$$

which tells us the net change in a differentiable function $F(x)$ *over an interval* $a \leq x \leq b$ *is the integral of its rate of change.*